# Axiomatic Solutions to a Simple Commons Problem

Geoffroy de Clippel*and Manipushpak Mitra†

March 31, 2006

PRELIMINARY NOTES - WORK IN PROGRESS
PLEASE DO NOT CIRCULATE

**Abstract**

The paper studies a simple commons problem, as in Moulin (2001). A set of agents collectively own a technology that produces many identical units of an indivisible object. Each agent needs and has use for only one unit of the object. A solution specifies how many units to produce, who should receive the object, and defines monetary transfers to compensate the agents that cannot consume the object. Simple axioms, adapted from the recent literature on queueing (see Maniquet, 2003), characterize a unique solution. It is the benefit analogue of the serial cost sharing rule already discussed in the literature, and is therefore called the serial surplus sharing rule. Assuming decreasing returns to scale, we also show that it coincides with the Shapley value of the coalitional form transferable utility game where the worth of a coalition is the efficient outcome in the absence of the complement coalition. We also develop a dual analysis, as Chun (2004) did for queueing problems.

## 1 Introduction

The importance of the problem of the commons has long been recognized (Dasgupta and Heal [2], Hardin [3], Moulin [7]). In this paper we address the issue of fair and efficient allocation in a simple commons problem. The framework for our problem is similar to that of Moulin [7]. In our simple commons problem, there is a finite set of agents who collectively own a technology that produces many identical units of an indivisible object. Each agent needs and has use for only one unit of the object. The decision problem, in the absence of monetary transfer, is binary that is whether or not to allocate the object to an agent

*Department of Economics, Rice Univeristy, 6100 Main Street, Houston TX 77005, USA. E-mail address: declippel@rice.edu.

†Economic Research Unit, Indian Statistical Institute 203, B. T. Road Kolkata-700108, India. E-mail address: mmitra@isical.ac.in.

given the technology and valuation of the agents. Agents have quasi-linear preferences. We try to identify the solutions to this allocation problem that are efficient and satisfy certain fairness properties by allowing for appropriate monetary transfers.

We first characterize the serial surplus sharing rule which is the benefit analogue of the serial cost sharing rule (Littlechild and Owen [4], Moulin and Shenker [8], [9]). The three axioms we use to characterize the serial surplus sharing rule are efficiency, equal treatment of equals and independent of higher valuations. While the first two axioms are very standard, the independence of higher valuations axiom is similar to the independence of preceding agents' impatience axiom, introduced by Maniquet [5] to characterize the Shapley value in the queueing problems with optimistic coalitional form transferable utility game. Assuming decreasing returns to scale, we then show that the serial surplus sharing rule is the Shapley value of the coalitional form transferable utility game where the worth of a coalition is the efficient outcome in the absence of the complement coalition. Finally, we apply a dual approach by replacing the independence of higher valuation axiom by the independence of lower valuations axiom. This gives us a rule, which we call the dual serial surplus sharing rule. The independence of lower valuations axiom is similar to the independence of following costs axiom introduced by Chun [1] to characterize the Shapley value in the queueing problems with pessimistic coalitional form transferable utility game. Assuming decreasing returns to scale, we also show that the dual serial surplus sharing rule is the Shapley value of the coalitional form transferable utility game where the worth of a coalition is the efficient outcome when all the non-members already used the technology.

The model is presented in section 2. The study of the serial surplus sharing rule is presented in section 3. The dual analysis is presented in section 4. These notes present partial results of a work in progress. Other questions that we are planning to address, or that we already addressed but are not included in these notes, are listed in section 5.

# 2 Model and Definitions

We consider a set $N$ of agents that collectively own a technology that allows to produce many (identical) units of an indivisible object. The production technology is described by its cost function $c : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$: $c(q)$ is the cost to produce $q$ units of the object, for each $q \in \mathbb{Z}_+$. We assume that producing nothing costs nothing: $c(0) = 0$. Each agent wants to consume at most one unit of the object. Let $v_i$ be the non-negative number that measures the satisfaction that agent $i$ derives from consuming the object. The vector $v \in \mathbb{R}_+^N$ is called the *vector of valuations*. Monetary transfers are feasible and utilities are quasi-linear. A *physical allocation* is a couple $(f, t) \in \{0, 1\}^N \times \mathbb{R}^N$, where $f$ determines the set of agents that get the object ($f(i) = 1$ if and only if agent $i$ receives one unit of the object), and $t$ is the vector of net transfers. It is *feasible* if $\sum_{i \in N} t_i \leq -c(\sum_{i \in N} f(i))$. Let $\mathcal{F}$ be the set of feasible physical allocation.

Agent $i$'s utility associated to a physical allocation $(f,t)$ [denoted by $u_i(f,t)$] equals $f(i)v_i + t_i$. A *utility profile* is a vector $x$ in $\mathbb{R}^N$. It is *feasible* if there exists $(f,t) \in \mathcal{F}$ such that $x_i = u_i(f,t)$ for each $i \in N$.

A *solution* is a function $\sigma : \mathbb{R}^N_+ \to \mathbb{R}^N$ that associates a feasible utility profile $\sigma(v)$ to each vector of valuations $v$. Reasonable solutions depend on the cost function and the solutions we define hereafter indeed satisfy this property. Nevertheless, all the axioms we will introduce are written for a given cost function and do not restrict the behavior of the solution when the cost function changes. For notational simplicity, the cost function does not appear explicitly as an argument of the solutions.

Non-empty subsets of $N$ are called *coalitions*. The set of coalitions will be denoted by $P(N)$. A *characteristic function* $\mathfrak{v}$ is a function that associates a real number to each coalition. The number $\mathfrak{v}(S)$ associated to a coalition $S$ is usually interpreted as the surplus that its members can share when they cooperate. The Shapley value is the most prominent normative solution defined for games in characteristic function. The value of an agent is a weighted sum of his marginal contributions to the different coalitions:

$$Sh_i(\mathfrak{v}) = \sum_{S \in P(N) \text{ s.t. } i \in S} \frac{(n-s)!(s-1)!}{n!}[\mathfrak{v}(S) - \mathfrak{v}(S \setminus \{i\})]$$

for each $i \in N$, where $n$ (resp. $s$) is the cardinality of $N$ (resp. $S$).

We conclude the section with some notations. Let $v$ be a vector of valuations and let $i$ be an agent. Then $N(v, \geq, i)$ denotes the set of agents whose valuation is greater or equal to $v_i$:

$$N(v, \geq, i) = \{j \in N | v_j \geq v_i\}.$$

The cardinality of $N(v, \geq, i)$ will be denoted by $n(v, \geq, i)$. Similar definitions apply for $\leq$, $>$, $<$, and $=$.

# 3 Serial Surplus Sharing

The objective of this section is to show how three simple axioms characterize a unique solution $\hat{\sigma}$ that can be computed via a simple formula (Proposition 1). Corollary 1 proposes an alternative way to compute the solution and justifies its name of serial surplus sharing. We also show in Corollary 2 that it coincides with the Shapley value of some characteristic function (with optimistic expectations), when the technology has decreasing returns to scale. The axioms go as follows.

**Efficiency** (EFF) $\sum_{i \in N} \sigma_i(v) = \max_{(f,t) \in \mathcal{F}} \sum_{i \in N} (f(i)v_i + t_i)$, for each $v \in \mathbb{R}^N_+$.

**Equal Treatment of Equals** (ETE) Let $v \in \mathbb{R}^N_+$, and let $(i,j) \in N \times N$. If $v_i = v_j$, then $\sigma_i(v) = \sigma_j(v)$.

**Independence of Higher Valuations** (IHV) Let $(v, v') \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ and let $i \in N$. If $v_i = v'_i$ and $v_j = v'_j$ for each $j \in N(v, <, i) \cup N(v', <, i)$, then $\sigma_i(v) = \sigma_i(v')$.

Efficiency imposes on the solution to exhaust the highest possible surplus. Equal treatment of equals is a minimal equity property. If two agents value the good identically, then they should get identical payoffs. Any anonymous solution satisfies ETE. The last axiom means that agent $i$'s payoff does not depend on how much agent $j$ values the object, if $j$ values it more than $i$.

**Proposition 1** *There exists a unique solution $\hat{\sigma}$ that satisfies EFF, ETE, and IHV. In addition, for each $v \in \mathbb{R}_+^N$ and each $i \in N$, we have:*

$$\hat{\sigma}_i(v) = \frac{\pi(v, i)}{n(v, \geq, i)} - \sum_{k \in N(v, <, i)} \frac{\pi(v, k)}{n(v, >, k) n(\geq, v, k)}, \tag{1}$$

*where*

$$\pi(v, k) = \max_{(f,t) \in \mathcal{F}} \sum_{j \in N} (f(j) \min\{v_j, v_k\} + t_j), \text{ for all } k$$

*is the highest surplus that the agents can create when they cooperate, if we assume that each agent's valuation is the minimum between $v_k$ and his original valuation.*

Proof: It is easy to check that $\hat{\sigma}$ defined in (1) satisfies both ETE and IHV. As for efficiency, we have to check that:

$$\sum_{i \in N} \frac{\pi(v, i)}{n(v, \geq, i)} - \sum_{i \in N} \sum_{k \in N(v, <, i)} \frac{\pi(v, k)}{n(v, >, k) n(\geq, v, k)} = \max_{(f,t) \in \mathcal{F}} \sum_{i \in N} (f(i) v_i + t_i),$$

for each $v \in \mathbb{R}_+^N$. Indeed, the coefficient of $\pi(v, i)$ on the left-hand side equals $\frac{1}{n(v, \geq, i)} - \sum_{j \in N(v, >, i)} \frac{1}{n(v, >, i) n(\geq, v, i)}$, for each $i \in N$. This coefficient is null, if $v_i < \max_{j \in N} v_j$, and equals $\frac{1}{n(v, =, i)}$, if $v_i = \max_{j \in N} v_j$. It is then easy to conclude, as $\pi(v, i) = \max_{(f,t) \in \mathcal{F}} \sum_{j \in N} (f(j) v_j + t_j)$ in the latter case.

Let now $\sigma$ be any solution that satisfies the three axioms. Let $v \in \mathbb{R}_+^N$, and let $(N_1, \ldots, N_K)$ be the unique partition of $N$ such that

1. If $i$ and $j$ belong to the same atom of the partition, then $v_i = v_j$;

2. If $i \in N_k$, $j \in N_{k'}$, and $k < k'$, then $v_i < v_j$.

We show that $\sigma(v) = \hat{\sigma}(v)$ by induction on $k$. Suppose first that $i \in N_1$. Consider $v'$, the constant vector of valuations defined as follows: $v'_j = v_i$, for each $j \in N$. We have:

$$\sigma_i(v) = \sigma_i(v') = \frac{\max_{(f,t) \in \mathcal{F}} \sum_{j \in N} (f(i) v'_j + t_j)}{n} = \frac{\pi(v, i)}{n} = \hat{\sigma}_i(v).$$

The first equality follows from IHV. The second equality follows from EFF and ETE (all the agents value the object identically in $v'$). The third and fourth equalities follow respectively from the definition of $\pi$ and the definition of $\hat{\sigma}$ in (1). Let now $k^* \in \{1, \ldots, K-1\}$. Suppose that $\sigma_i(v) = \hat{\sigma}_i(v)$, for each $i \in \cup_{k=1}^{k^*} N_k$, and let $i \in N_{k^*+1}$. Consider $v''$, the vector of valuations defined as follows: $v''_j = v_j$, for each $j \in N(v, <, i)$, and $v''_j = v_i$, for each $j \in N(v, \geq, i)$. The induction hypothesis and IHV (applied to both $\sigma$ and $\hat{\sigma}$) imply that $\sigma_j(v'') = \sigma_j(v) = \hat{\sigma}_j(v) = \hat{\sigma}_j(v'')$, for each $j \in N(v, <, i)$. On the other hand, all the agents in $N(v, \geq, i)$ value the object identically. Hence, ETE and EFF (applied to both $\sigma$ and $\hat{\sigma}$) imply that $\sigma(v'') = \hat{\sigma}(v'')$. IHV (applied to both $\sigma$ and $\hat{\sigma}$) imply that $\sigma_i(v) = \hat{\sigma}_i(v)$. ∎

The next corollary offers an alternative formula to compute $\hat{\sigma}$. It also explains the title of the section.

**Corollary 1** *Let us assume without loss of generality that* $v_1 \leq \ldots \leq v_n$. *Then*

$$\hat{\sigma}_i(v) = \sum_{j=1}^{i} \frac{\pi(v,j) - \pi(v,j-1)}{n(\geq, v, j)},$$

*for each* $i \in N$, *with the convention that* $\pi(v, 0) = 0$.

Let $S$ be a coalition. A *physical allocation for* $S$ is a couple $(f, t) \in \{0,1\}^S \times \mathbb{R}^S$, where $f$ determines the set of agents that get the object, and $t$ is the vector of net transfers. It is *feasible for* $S$, assuming that its members may use the technology before the other agents, if $\sum_{i \in S} t_i \leq -c(\sum_{i \in S} f(i))$. Let $\mathcal{F}(S)$ be the set of physical allocations that are feasible for $S$ in this sense. A technology $c$ has *decreasing returns to scale* if $c(q+1) - c(q)$ does not decrease with $q$.

**Corollary 2** *Consider a technology with decreasing returns to scale. Then, for each* $v \in \mathbb{R}_+^N$,

$$\hat{\sigma}(v) = Sh(\hat{\mathfrak{v}}),$$

*where*

$$\hat{\mathfrak{v}}(S) = \max_{(f,t) \in \mathcal{F}(S)} \sum_{j \in S} (f(j) v_j + t_j),$$

*for each coalition* $S$.

Proof: It is easy to check that the solution $Sh(\hat{\mathfrak{v}})$ satisfies EFF and ETE. We show that it also satisfies IHV. The result then follows from Proposition 1. We need the following lemma. It shows that the optimal allocations for each coalition can be obtained by comparing the marginal costs and benefits of producing an additional unit of the object, provided that the technology has decreasing returns to scale.

**Lemma 1** *Let $S$ be a coalition, and let $\tau_S : S \to \{1, \ldots, s\}$ be a bijection such that*

$$\tau_S(i) \geq \tau_S(j) \Rightarrow v_i \geq v_j,$$

*for all $(i,j) \in S \times S$. If $c(1) \geq \max_{i \in S} v_i$, then $\hat{\mathfrak{v}}(S) = 0$. If $c(1) < \max_{i \in S} v_i$, then*

$$\hat{\mathfrak{v}}(S) = \sum_{k=1}^{q(S)} v_{\tau_S^{-1}(k)} - c(q(S)),$$

*where $q(S) = \max\{k \in \{1, \ldots, s\} | c(k) - c(k-1) < v_{\tau_S^{-1}(k)}\}$.*

Proof: Let $\hat{\mathfrak{v}}(S, 0) = 0$, and

$$\hat{\mathfrak{v}}(S, q) = \max_{(f,t) \in \mathcal{F}(S) \text{ s.t. } \sum_{j \in S} f(j) = q} \sum_{i \in S} (f(i) v_i + t_i),$$

for each $q \in \{1, \ldots, s\}$. It is easy to check that

$$\hat{\mathfrak{v}}(S, q) = \sum_{k=1}^{q} v_{\tau_S^{-1}(k)} - c(q),$$

for each $q \in \{1, \ldots, s\}$. Hence the sequence of numbers obtained by varying $q$ is characterized by the following recursive equation

$$(\forall q \in \{1, \ldots, s\}) : \hat{\mathfrak{v}}(S, q) = \hat{\mathfrak{v}}(S, q-1) + v_{\tau_S^{-1}(q)} - [c(q) - c(q-1)].$$

On the other hand,

$$\hat{\mathfrak{v}}(S) = \max_{q \in \{0, \ldots, s\}} \hat{\mathfrak{v}}(S, q).$$

The lemma then follows from the fact that $v_{\tau_S^{-1}(q)} - [c(q) - c(q-1)]$ does not increase when $q$ increases (remember that the technology has decreasing returns to scale). ■

$Sh(\hat{\mathfrak{v}})$ satisfies IHV: Let $i$, $v$, and $v'$ be defined as in IHV. Let $\hat{\mathfrak{v}}$ and $\hat{\mathfrak{v}}'$ be the characteristic functions associated to $v$ and $v'$ respectively. Let $S$ be a coalition that does not contain $i$. We show that $\hat{\mathfrak{v}}(S \cup \{i\}) - \hat{\mathfrak{v}}(S) = \hat{\mathfrak{v}}'(S \cup \{i\}) - \hat{\mathfrak{v}}'(S)$. Observe that $S \cap N(v, <, i) = S \cap N(v', <, i)$. Let $T$ be this set. Let $\tau_S : S \to \{1, \ldots, s\}$ and $\tau'_S : S \to \{1, \ldots, s\}$ be two bijections such that

$$(\forall (j,k) \in S \times S) : \tau_S(j) \geq \tau_S(k) \Rightarrow v_j \geq v_k,$$

$$(\forall (j,k) \in S \times S) : \tau'_S(j) \geq \tau'_S(k) \Rightarrow v'_j \geq v'_k,$$

and

$$(\forall j \in T) : \tau_S(j) = \tau'_S(j)$$

There exist two such bijections because $v_j = v'_j$ for each $j \in T$. Let now $\tau_{S\cup\{i\}} : S \cup \{i\} \to \{1, \ldots, s+1\}$ and $\tau'_{S\cup\{i\}} : S \cup \{i\} \to \{1, \ldots, s+1\}$ be the two bijections defined as follows:

$$\tau_{S\cup\{i\}}(j) = \begin{cases} \tau_S(j) & \text{for each } j \in S \setminus T \\ \tau_S(j) + 1 & \text{for each } j \in T \\ s - t + 1 & \text{if } j = i, \end{cases}$$

$$\tau'_{S\cup\{i\}}(j) = \begin{cases} \tau'_S(j) & \text{for each } j \in S \setminus T \\ \tau'_S(j) + 1 & \text{for each } j \in T \\ s - t + 1 & \text{if } j = i. \end{cases}$$

Observe that

$$(\forall (j,k) \in (S \cup \{i\}) \times (S \cup \{i\})) : \tau_{S\cup\{i\}}(j) \geq \tau_{S\cup\{i\}}(k) \Rightarrow v_j \geq v_k,$$

$$(\forall (j,k) \in (S \cup \{i\}) \times (S \cup \{i\})) : \tau'_{S\cup\{i\}}(j) \geq \tau'_{S\cup\{i\}}(k) \Rightarrow v'_j \geq v'_k.$$

Lemma 1 implies that

$$\hat{\mathfrak{v}}(S \cup \{i\}) - \hat{\mathfrak{v}}(S) = \sum_{k=1}^{q(S\cup\{i\})} v_{\tau^{-1}_{S\cup\{i\}}(k)} - \sum_{k=1}^{q(S)} v_{\tau_S^{-1}(k)} + c(q(S)) - c(q(S \cup \{i\})),$$

where $q(S) = \max\{k \in \{1, \ldots, s\} | c(k) - c(k-1) < v_{\tau_S^{-1}(k)}\}$ and $q(S \cup \{i\}) = \max\{k \in \{1, \ldots, s+1\} | c(k) - c(k-1) < v_{\tau^{-1}_{S\cup\{i\}}(k)}\}$. Observe that $\tau_{S\cup\{i\}}(i) = \tau'_{S\cup\{i\}}(i)$. Let $k$ be this number. If $c(k) - c(k-1) \geq v_i$, then $q(S) = q(S \cup \{i\}) \leq s - t$. Hence $\hat{\mathfrak{v}}(S \cup \{i\}) - \hat{\mathfrak{v}}(S) = 0$. Similarly, $\hat{\mathfrak{v}}'(S \cup \{i\}) - \hat{\mathfrak{v}}'(S) = 0$, and we are done. If $c(k) - c(k-1) < v_i$, then $q(S) \geq s - t$ and $q(S \cup \{i\}) \geq s - t + 1$. Hence,

$$\hat{\mathfrak{v}}(S \cup \{i\}) - \hat{\mathfrak{v}}(S) = \sum_{k=s-t+1}^{q(S\cup\{i\})} v_{\tau^{-1}_{S\cup\{i\}}(k)} - \sum_{k=s-t+1}^{q(S)} v_{\tau_S^{-1}(k)} + c(q(S)) - c(q(S \cup \{i\})).$$

A similar reasoning implies that $q'(S) \geq s - t$, $q'(S \cup \{i\}) \geq s - t + 1$, and

$$\hat{\mathfrak{v}}'(S \cup \{i\}) - \hat{\mathfrak{v}}'(S) = \sum_{k=s-t+1}^{q'(S\cup\{i\})} v'_{\tau^{-1}_{S\cup\{i\}}(k)} - \sum_{k=s-t+1}^{q'(S)} v'_{\tau_S^{-1}(k)} + c(q'(S)) - c(q'(S \cup \{i\})).$$

Finally, $v_j = v'_j$, for each $j \in T$, implies that $q(S) = q'(S)$, $q(S \cup \{i\}) = q'(S \cup \{i\})$, and hence $\hat{\mathfrak{v}}(S \cup \{i\}) - \hat{\mathfrak{v}}(S) = \hat{\mathfrak{v}}'(S \cup \{i\}) - \hat{\mathfrak{v}}'(S)$. ■

# 4 A Dual Approach

We propose a dual version of IHV and show that there exits a unique solution that satisfies EFF, ETE and this new axiom (Proposition 2). We give two formula to compute it (Proposition 2 and Corollary 3). We also show in Corollary

4 that it coincides with the Shapley value of some characteristic function (with pessimistic expectations), when the technology has decreasing returns to scale.

**Independence of Lower Valuations** (ILV) Let $(v, v') \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ and let $i \in N$. If $v_i = v'_i$ and $v_j = v'_j$ for each $j \in N(v, >, i) \cup N(v', >, i)$, then $\sigma_i(v) = \sigma_i(v')$.

Agent $i$'s payoff does not depend on how much agent $j$ values the object, if $j$ values it less than $i$.

**Proposition 2** *There exists a unique solution $\hat{\sigma}^*$ that satisfies EFF, ETE, and ILV. In addition, for each $v \in \mathbb{R}_+^N$ and each $i \in N$, we have:*

$$\hat{\sigma}_i^*(v) = \frac{\pi^*(v, i)}{n(v, \leq, i)} - \sum_{k \in N(v, >, i)} \frac{\pi^*(v, k)}{n(v, <, k) n(\leq, v, k)}, \tag{2}$$

*where*

$$\pi^*(v, k) = \max_{(f,t) \in \mathcal{F}} \sum_{j \in N} (f(j) \max\{v_j, v_k\} + t_j), \text{ for all } k$$

*is the highest surplus that the agents can create when they cooperate, if we assume that each agent's valuation is the maximum between $v_k$ and his original valuation.*

The next corollary offers an alternative formula to compute $\hat{\sigma}$. It also explains the title of the section.

**Corollary 3** *Let us assume without loss of generality that $v_1 \geq \ldots \geq v_n$. Then*

$$\hat{\sigma}_i^*(v) = \sum_{j=1}^{i} \frac{\pi^*(v, j) - \pi^*(v, j+1)}{n(\leq, v, j)},$$

*for each $i \in N$, with the convention that $\pi(v, n+1) = 0$.*

Let $S$ be a coalition. A physical allocation $(f, t)$ for $S$ *star-feasible* if $\sum_{i \in S} t_i \leq -c(n - s + \sum_{i \in S} f(i))$. Let $\mathcal{F}^*(S)$ be the set of physical allocations that are star-feasible for $S$ in this sense.

**Corollary 4** *Consider a technology with decreasing returns to scale. Then, for each $v \in \mathbb{R}_+^N$,*

$$\hat{\sigma}^*(v) = Sh(\hat{\mathfrak{v}}^*),$$

*where*

$$\hat{\mathfrak{v}}^*(S) = \max_{(f,t) \in \mathcal{F}^*(S)} \sum_{j \in S} (f(j) v_j + t_j),$$

*for each coalition $S$.*

## 5 Work in Progress

1. Study of applications, e.g. allocation of one or many free good(s) (particular case of Moulin [6] and the study of queueing problems with identical waiting costs, but different levels of satisfaction when the job is executed.

2. Present a new point of view that unify the two solutions $\hat{\sigma}$ and $\hat{\sigma}^*$. We weaken IHV and ILV to define a larger class of solutions $\Sigma$ that includes both $\hat{\sigma}$ and $\hat{\sigma}^*$. If $\geq^E$ represents the lexicographic egalitarian social welfare ordering, then $\hat{\sigma}^* \geq^E \sigma \geq^E \hat{\sigma}$, for each $\sigma \in \Sigma$. This shows that $\hat{\sigma}$ and $\hat{\sigma}^*$ are extreme solutions within $\Sigma$.

3. Compare $\hat{\sigma}$ and $\hat{\sigma}^*$ with other solutions, including competitive equilibria with equal income and the virtual price solutions.

4. Study of the random ownership game for the allocation of free goods.

5. Examples to highlight the importance of the decreasing returns to scale in Corollary 2 and 4.

6. Study of the non-cooperative implementation of $\hat{\sigma}$ and $\hat{\sigma}^*$.

## References


[1] Chun Y., 2004. A Note on Maniquet's Characterizations of the Shapley Value in Queueing Problems. *mimeo* University of Rochester.

[2] Dasgupta P., Heal G., 1979. Economic Theory and Exhaustible Resources. Cambridge University Press, Cambridge, MA.

[3] Hardin G., 1968. The tragedy of the commons. Science 162, 1243-1248.

[4] Littlechild S. C., Owen G., 1973. A Simple Expression of the Shapley Value in a Special Case. Management Science 20, 99-107.

[5] Maniquet F., 2003. A characterization of the Shapley value in queueing problems. Journal of Economic Theory 109, 90-103.

[6] Moulin H., 1992. An Application of the Shapley value to Fair Division with Money. Econometrica 60, 1331-1349.

[7] Moulin H., 2001. Three solutions to a simple commons problem. Seoul Journal of Economics 14, 245-267.

[8] Moulin H., Shenker S., 1992. Serial Cost Sharing. Econometrica 60, 1009-1037.

[9] Moulin H., Shenker S., 1994. Average Cost Pricing versus Serial Cost Sharing: An Axiomatic Comparison. Journal of Economic Theory 64, 178-201.